

# $L^p$ ESTIMATES FOR FRACTIONAL SCHRÖDINGER OPERATORS WITH KATO CLASS POTENTIALS

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ABSTRACT. Let  $\alpha > 0$ ,  $H = (-\Delta)^\alpha + V(x)$ ,  $V(x)$  belongs to the higher order Kato class  $K_{2\alpha}(\mathbb{R}^n)$ . For  $1 \leq p \leq \infty$ , we prove a polynomial upper bound of  $\|e^{-itH}(H + M)^{-\beta}\|_{L^p, L^p}$  in terms of time  $t$ . Both the smoothing exponent  $\beta$  and the growth order in  $t$  are almost optimal compared to the free case. The main ingredients in our proof are pointwise heat kernel estimates for the semigroup  $e^{-tH}$ . We obtain a Gaussian upper bound with sharp coefficient for integral  $\alpha$  and a polynomial decay for fractal  $\alpha$ .

## 1. INTRODUCTION

Let  $\alpha > 0$ , consider the following fractional Schrödinger equation

$$(1.1) \quad i\partial_t u + (-\Delta)^\alpha u + V(x)u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n,$$

with initial condition

$$(1.2) \quad u(0, x) = u_0(x).$$

Here  $V(x)$  belongs to the higher order Kato class  $K_{2\alpha}(\mathbb{R}^n)$  (see the Definition 1.1), the fractional Laplacian  $(-\Delta)^\alpha$  is defined as a Fourier multiplier with symbol  $|\xi|^{2\alpha}$ . For  $\alpha \in (0, 1)$ , equation (1.1)-(1.2) was introduced by Laskin [30] as a result of extending the Feynman path integral, from the Brownian-like to Levy-like quantum mechanical paths. Recently, the fractional Schrödinger equation has been studied by several authors in the literatures. We refer the readers to [28] for scattering theory and [13, 22, 24] for well posedness and ill posedness of nonlinear equations, respectively.

In these papers as well as the most existing references on Schrödinger equation, one often chooses  $L^2$ -based Sobolev spaces as the working space. The main reason lies in that the solution of

$$(1.3) \quad i\partial_t u = \Delta u, \quad u(0, x) = u_0(x)$$

may not be bounded in  $L^p(\mathbb{R}^n)$  if the initial data  $u_0$  only belongs to  $L^p(\mathbb{R}^n)$ ,  $p \neq 2$ , see e.g. [23]. Nevertheless, the solution of (1.3) will be bounded in  $L^p(\mathbb{R}^n)$  if we impose further some smoothness assumptions on the data. Indeed, Brenner, Thomée and Wahlbin obtained (see [10], p.134) the following estimate

$$(1.4) \quad c(1 + |t|)^{n_p} \leq \|e^{-it\Delta}(1 - \Delta)^{-\beta}\|_{L^p, L^p} \leq C(1 + |t|)^{n_p}, \quad t \in \mathbb{R},$$

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*Date:* November 7, 2015.

*2000 Mathematics Subject Classification.* Primary 43A15, 35K08; Secondary 35Q41.

*Key words and phrases.*  $L^p$  estimates, heat kernel estimates, fractional Schrödinger equations.

for  $1 < p < \infty$  and  $\beta \geq n_p$ , here and below

$$n_p = n \left| \frac{1}{p} - \frac{1}{2} \right|.$$

The smoothing exponent  $\beta$  is sharp in sense that  $\|e^{-it\Delta}(1 - \Delta)^{-\beta}\|_{L^p, L^p}$  is unbounded if  $t \neq 0$  and  $\beta < n_p$ . More generally, for  $\alpha > 0$ , it was shown by Fefferman and Stein [19] and Miyachi [31] that

$$(1.5) \quad \|e^{it(-\Delta)^\alpha}(1 + (-\Delta)^\alpha)^{-\beta}\|_{L^p, L^p} \leq C(1 + |t|)^{n_p}, \quad t \in \mathbb{R}$$

holds for  $1 < p < \infty$  and  $\beta \geq n_p$ . It's interesting to note that  $\beta \geq n_p$  in (1.5) can be relaxed to  $\beta \geq (n-1)|\frac{1}{p} - \frac{1}{2}|$  if  $\alpha = \frac{1}{2}$ , see [31]. In the presence of a potential, Jensen and Nakamura [25, 26] proved the following almost optimal estimate

$$(1.6) \quad \|e^{-it(-\Delta+V)}(-\Delta+V+M)^{-\beta}\|_{L^p, L^p} \leq C(1 + |t|)^\gamma, \quad t \in \mathbb{R}$$

where  $1 \leq p \leq \infty$ ,  $\beta, \gamma > n_p$ ,  $M$  is a sufficiently large constant,  $V \in K_2(\mathbb{R}^n)$ . The restriction  $\beta > n_p$  is necessary for (1.6) to hold in the case  $p = 1$  or  $\infty$ .

In order to state our main result, we need to recall the definition of higher order Kato class firstly, see e.g. [40].

**Definition 1.1.** Let  $\alpha > 0$ , and set

$$\omega_\alpha(x) = \begin{cases} |x|^{2\alpha-n}, & \text{if } 0 < 2\alpha < n, \\ \ln|x|^{-n}, & \text{if } 2\alpha = n, \\ 1, & \text{if } 2\alpha > n. \end{cases}$$

A real-valued measurable function  $V(x)$  on  $\mathbb{R}^n$  is said to lie in  $K_{2\alpha}(\mathbb{R}^n)$  if

$$\lim_{\delta \rightarrow 0} \sup_{x \in \mathbb{R}^n} \int_{|x-y| < \delta} \omega_\alpha(x-y)|V(y)|dy = 0, \quad \text{for } 0 < 2\alpha \leq n,$$

and

$$\sup_{x \in \mathbb{R}^n} \int_{|x-y| < 1} |V(y)|dy < \infty, \quad \text{for } 2\alpha > n.$$

By Definition 1.1, the class  $K_2(\mathbb{R}^n)$  coincides with the classical Kato class on  $\mathbb{R}^n$ . The main result in this paper reads as follows.

**Theorem 1.2.** Let  $\alpha > 0$ ,  $H = (-\Delta)^\alpha + V(x)$ ,  $V \in K_{2\alpha}(\mathbb{R}^n)$ , then for some large enough constant  $M$ , the estimate

$$(1.7) \quad \|e^{-itH}(H+M)^{-\beta}\|_{L^p, L^p} \leq C(1 + |t|)^\gamma, \quad t \in \mathbb{R},$$

holds for  $1 \leq p \leq \infty$ ,  $\beta, \gamma > n|\frac{1}{2} - \frac{1}{p}|$ ,  $C$  is a positive constant independent of  $t$ .

Theorem 1.2 generalizes estimate (1.6) of the second order Schrödinger equation to higher order cases inspired by (1.5). The smoothing exponent  $\beta$  and the growth order  $\gamma$  are almost optimal. In the case  $V(x) = 0$ , since  $e^{-itH}(H+M)^{-\beta}$  is a Fourier multiplier, one can obtain (1.7) by  $L^p$  multiplier criterion and interpolation technique, see [10, 19, 31] for details. The approach does not work anymore for general  $V(x) \in K_{2\alpha}(\mathbb{R}^n)$ . There is another strategy to obtain 1.7. In fact, if  $e^{zH}$  is a bounded analytic semigroup on  $L^2$ ,  $\Re z > 0$ , satisfying the estimate

$$(1.8) \quad |K(t, x, y)| \leq C_1 t^{-\frac{n}{2m}} \exp \left\{ -C_2 \frac{|x-y|^{\frac{2m}{2m-1}}}{t^{\frac{1}{2m-1}}} + C_3 t \right\}, \quad t > 0,$$

where  $C_1, C_2, C_3$  are positive constants, the integral kernel is given by

$$(e^{tH}f)(x) = \int_{\mathbb{R}^n} K(t, x, y)f(y)dy, \quad f \in L^2(\mathbb{R}^n),$$

then (1.7) holds. The result was proved partially by Zheng and Zhang [39], fully by Carron, Coulhon and Ouhabaz [11]. Thus, the proof of Theorem 1.2 is reduced to establish the kernel estimate of the semigroup  $e^{-t((-\Delta)^\alpha + V)}$ . It turns out that the kernel estimates are very different if  $\alpha$  is an integer or not. Therefore, we divide our discussion into two sections.

In section 2, we treat the kernel estimates when  $\alpha = m$  is a positive integer, and prove the following quantitative bounds.

**Theorem 1.3.** *Assume  $V \in K_{2m}$ , then for any  $0 < \varepsilon \ll 1$ , there exists some constants  $C, V_{\varepsilon^2}$  depending on  $V$  and  $\varepsilon$  (see Section 2 for the definition), such that the kernel of the semigroup  $e^{-t((-\Delta)^m + V)}$  satisfies*

$$|K(t, x, y)| \leq C\varepsilon^{-(n+1)([\frac{n}{2m}]+3)}t^{-\frac{n}{2m}} \exp \left\{ -\varsigma_m \frac{|x-y|^{\frac{2m}{2m-1}}}{(1+\varepsilon)t^{\frac{1}{2m-1}}} + \varepsilon^{-2}V_{\varepsilon^2}t \right\}, \quad t > 0,$$

where  $[s]$  denotes the integer part of  $s$ ,  $\varsigma_m = (2m-1)(2m)^{-\frac{2m}{2m-1}} \sin\left(\frac{\pi}{4m-2}\right)$ .

Theorem 1.3 recovers the classical result in the case  $m = 1$ , see Simon [35]. The proof in [35] relies heavily on the famous Feynman-Kac path formula, which is based on the deep connection between the second order elliptic operator and probability theory. However, no such connection is available in the higher order case. Zheng and Yao [40] tried to overcome this difficulty by purely semigroup approach, but only  $L^p - L^q$  estimates are obtained. Recently, Deng, Ding, and Yao [17] established (1.8) for the kernel of  $e^{-t((-\Delta)^m + V)}$  by Davies perturbation strategy [14]. Compared to the work [17], Theorem 1.3 gives quantitative information on the constants appearing in (1.8). In particular, the sharp constant  $\varsigma_m$  is obtained. The number  $\varsigma_m$  is first deduced by Barbatis and Davies in [6]. They proved that the kernel of free heat semigroup  $e^{-t(-\Delta)^m}$  satisfies

$$(1.9) \quad |K(t, x, y)| \leq C \exp \left\{ -\varsigma_m \frac{|x-y|^{\frac{2m}{2m-1}}}{rt^{\frac{1}{2m-1}}} \right\}, \quad t > 0$$

with  $r > 1$ . Later, Dungey [18] improved (1.9) to  $r = 1$ . The heat kernel estimates with sharp constant  $\varsigma_m$  of elliptic operators of order  $2m$  were obtained by Barbatis [3, 4, 5]. But the restriction  $2m > n$  is need there for technique reasons.

In order to obtain the sharp constant  $\varsigma_m$ , we borrow the idea from Barbatis and Davies [6] to consider the semigroup generated by the conjugated operator  $P_\lambda(D) = e^{-\lambda\phi}(-\Delta)^m e^{\lambda\phi}$  with  $\phi$  being linear. This differs from the treatment in [17], where  $\phi$  was chosen to be bounded smoothing functions. The choice of linear weight allows us to obtain more precise  $L^1 - L^1$  resolvent estimates of  $(\mu + P_\lambda(D))^{-1}$  by taking full advantage of pointwise kernel bounds of  $e^{-z(-\Delta)^m}$  with “complex time”  $\Re z > 0$ . On the other hand, since the sum  $P_\lambda(D) + V$  can’t be understood in the operator sense on  $L^2(\mathbb{R}^n)$ , we construct a sesquilinear form  $Q_{\lambda,V}$  to overcome this difficulty. After that, we shall show that the semigroup constructed on  $L^1(\mathbb{R}^n)$  and on  $L^2(\mathbb{R}^n)$  are consistent. Finally, we prove a quantitative  $L^1 - L^\infty$  type estimate of the heat semigroup  $e^{-t(P_\lambda(D)+V)}$  by its smoothing effect, which implies Theorem 1.3 immediately.

In section 3, we are devoted to the heat kernel estimates of  $e^{-t((-\Delta)^\alpha + V)}$  for fractional  $\alpha > 0$ . Since the symbol  $e^{-t|\xi|^{2\alpha}}$  is not smooth, the exponential decay of the kernel of the free semigroup  $e^{-t((-\Delta)^\alpha)}$  is not expected. Our result is contained in the following

**Theorem 1.4.** *Assume  $V \in K_{2\alpha}(\mathbb{R}^n)$ ,  $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$ , then for any  $0 < \varepsilon \ll 1$ , there exists some constants  $C$ , and  $\mu_{\varepsilon, V}$  depending on  $V$  and  $\varepsilon$ , such that the kernel of the semigroup  $e^{-t((-\Delta)^\alpha + V)}$  satisfies*

$$(1.10) \quad |K(t, x, y)| \leq Ce^{\mu_{\varepsilon, V}t} \frac{t}{(|x - y|^2 + t^{\frac{1}{\alpha}})^{\frac{n}{2} + \alpha}}, \quad t > 0.$$

In the free case  $V(x) = 0$ , the same bounds were obtained by Balabane and Emami-Rad [2], Miyajima and Shindoh [32]. They also showed that the decay index  $\frac{n}{2} + \alpha$  in the upper bounds is sharp as  $|x - y|$  goes to infinity. In the case  $0 < \alpha < 1$ , Theorem 1.4 is contained in Proposition 4.1 of Shindoh [34], though the fact is not pointed out explicitly there. The proof in [34] is based on Voigt's theory of absorption semigroup [36, 37]. It only works for positivity semigroups and thus fails in the case  $\alpha > 1$ . In [8], Bogdan and Jackubowski proved the estimate (1.10) for the kernel of the semigroup generated by  $(-\Delta)^\alpha + b(x) \cdot \nabla$  for  $\frac{1}{2} < \alpha < 1$ ,  $V \in K_1(\mathbb{R}^n)$ . Xie and Zhang [38] improved the result to  $\alpha = \frac{1}{2}$ . The main tools in [8] are the so-called the 3P Theorem and a characterization of Kato class potentials  $K_{2\alpha}(\mathbb{R}^n)$ . In section 3, we adapt the tools to the case  $\alpha > 1$  and prove Theorem 1.4. Our new ingredient in this part is to show that  $-((-\Delta)^\alpha + V)$  is the generator of the constructed semigroup in usual sense, while only weak generator was obtained in [8, 38]. To this end, we shall exploit the analyticity of the semigroup generated by  $-(-\Delta)^\alpha$  and the uniqueness of vector-valued Laplace transform.

In section 4, we prove Theorem 1.2. If  $\alpha = m$  is an integer, then Theorem 1.2 follows from Theorem 1.3 directly as said before. However, if  $\alpha$  is fractional, Theorem 1.4 gives only polynomial decay in  $|x - y|$  of the kernel as  $|x - y|$  goes to infinity. Thus the abstract approach based on kernel estimates does not work. Fortunately, the distinct behavior of the heat kernels in two cases lead to the same form of uniform resolvent estimates from  $L^p$  to the amalgam space  $l^p(L^q)$  (see Theorem 4.1). This fact allows us to prove Theorem 1.2 in a unified manner. Inspired by the work Jensen and Nakamura [25, 26], we shall show that the operator  $e^{it\theta H}g(\theta H)$  is bounded in  $L^p$  with  $p \geq 1$ , uniformly for  $\theta \in (0, 1]$  and  $g$  in bounded subsets of  $C_0^\infty(\mathbb{R})$ , which in turn implies Theorem 1.2 after a dyadic decomposition.

## 2. SHARP HEAT KERNEL ESTIMATES FOR THE INTEGER CASE

Let  $\mathcal{L}^{2m,1}(\mathbb{R}^n)$  be the  $2m$  order Bessel space in  $L^1(\mathbb{R}^n)$ . For  $V \in K_{2m}$ , it follows from [40] that, for any  $\varepsilon > 0$ , there exists  $\sigma_\varepsilon$  such that

$$\|V\varphi\|_{L^1} \leq \varepsilon \|(-\Delta)^m \varphi\|_{L^1} + \sigma_\varepsilon \|\varphi\|_{L^1}, \quad \text{for all } \varphi \in \mathcal{L}^{2m,1}(\mathbb{R}^n).$$

For our purpose, fixed  $\varepsilon \in (0, 1)$ , we define a number

$$V_\varepsilon = \inf \{ \sigma : \sigma \in E_\varepsilon \}$$

where

$$E_\varepsilon = \left\{ \sigma \geq 0 : \|V\varphi\|_{L^1} \leq \varepsilon \|(-\Delta)^m \varphi\|_{L^1} + \sigma \|\varphi\|_{L^1}, \quad \text{for all } \varphi \in \mathcal{L}^{2m,1}(\mathbb{R}^n) \right\}.$$

It's easy to see that  $E$  is a non-empty, bounded from below, connected closed set, thus  $V_\varepsilon$  is well defined.

Let  $a \in \mathbb{R}^n$ ,  $|a| = 1$ ,  $\lambda > 0$ . Consider the following operator

$$P_\lambda(D) = e^{-\lambda a \cdot x} (-\Delta)^m e^{\lambda a \cdot x}$$

with maximal domain in  $L^1(\mathbb{R}^n)$ . This is a partial differential operator with constant coefficients though the spacial variable  $x$  get involved in the expression. In fact, it is easy to check that

$$P_\lambda(D) = (-\Delta - \lambda^2 + 2i\lambda a \cdot D)^m$$

with  $D = (\frac{1}{i} \frac{\partial}{\partial x_1}, \dots, \frac{1}{i} \frac{\partial}{\partial x_n})$ , and its symbol  $P_\lambda(\xi)$  satisfies the following property (see [6])

$$(2.1) \quad \Re P_\lambda(\xi) \geq -b_m \lambda^{2m}, \quad \forall \xi \in \mathbb{R}^n$$

where  $b_m = (\sin \frac{\pi}{4m-2})^{-(2m-1)}$ . Here and below,  $\Re f$  denotes the real part of  $f$ .

Denote by  $H_{\lambda,V} = P_\lambda(D) + V$ ,  $P(D) = (-\Delta)^m$ .

**Theorem 2.1.** *The operator  $-H_{\lambda,V}$  generates an analytic semigroup in  $L^1(\mathbb{R}^n)$ , the semigroup satisfies*

$$\|e^{-tH_{\lambda,V}}\|_{L^1, L^1} + \|tH_{\lambda,V}e^{-tH_{\lambda,V}}\|_{L^1, L^1} \leq C\varepsilon^{-(n+1)} \exp\{t(\varepsilon^{-2}V_{\varepsilon^2} + b_m(1+\varepsilon)\lambda^{2m})\},$$

for  $t > 0$ ,  $0 < \varepsilon < c$ , where  $c, C$  are some small and large enough constants, respectively.

*Proof.* We divide the proof into three steps.

**Step 1.** The estimates of  $(\mu + P_\lambda(D))^{-1}$ . The resolvent can be understood as an integral operator, more precisely

$$(\mu + P_\lambda(D))^{-1}\varphi = h(\cdot, \mu) * \varphi(\cdot),$$

where  $h(x, \mu) = \int_0^\infty \int_{\mathbb{R}^n} e^{ix \cdot \xi} e^{-z(\mu + P_\lambda(\xi))} d\xi dz$ ,  $*$  denotes the convolution. It has been proved [6, 18] that, for  $t > 0$

$$(2.2) \quad \left| \int_{\mathbb{R}^n} e^{ix \cdot \xi} e^{-t|\xi|^{2m}} d\xi \right| \leq Ct^{-n/2m} \exp \left\{ -\varsigma_m |x|^{\frac{2m}{2m-1}} / t^{\frac{1}{2m-1}} \right\}.$$

From this, similar to ([39], Theorem 2.1), we have

$$(2.3) \quad \left| \int_{\mathbb{R}^n} e^{ix \cdot \xi} e^{-z|\xi|^{2m}} d\xi \right| \leq C|\Re z|^{-n/2m}, \quad \Re z > 0.$$

Using (2.2) and (2.3), repeating word by word of the proof of Lemma 9 in [14], we obtain

$$(2.4) \quad \left| \int_{\mathbb{R}^n} e^{ix \cdot \xi} e^{-z|\xi|^{2m}} d\xi \right| \leq C|z|^{-n/2m} \exp \left\{ -(1-\varepsilon)\varsigma_m |x|^{\frac{2m}{2m-1}} / |z|^{\frac{1}{2m-1}} \right\}$$

for  $\Re z > 0$ ,  $|\arg z| \leq \theta$ ,  $\theta = \arctan(\varepsilon/C)$ ,  $0 < \varepsilon < 1$ . Changing variable  $\xi \rightarrow \xi - i\lambda a$ , and shifting the path of integration in the definition of  $h$  yields that

$$h(x, \mu) = e^{\lambda a \cdot x} \int_0^\infty \int_{\mathbb{R}^n} e^{ix \cdot \xi} \exp \left\{ -(\mu + |\xi|^{2m}) \rho e^{i\theta \operatorname{sgn}(\operatorname{Im} \mu)} \right\} d\xi d\rho,$$

where  $\operatorname{sgn}(\cdot)$  denotes the standard sign function. Thus, using (2.4), Fubini's theorem and Young's inequality, we deduce that

$$\begin{aligned}
& \|(\mu + P_\lambda(D))^{-1}\|_{L^1, L^1} \leq \|h(\cdot, \mu)\|_{L^1} \\
& \leq \int_{\mathbb{R}^n} dx e^{\lambda a \cdot x} \int_0^\infty d\rho C e^{-(\Re \mu)\rho \cos \theta - |\operatorname{Im} \mu| \rho \sin \theta} \rho^{-n/2m} \\
& \quad \cdot \exp \left\{ -(1 - \varepsilon) \varsigma_m |x|^{\frac{2m}{2m-1}} / \rho^{\frac{1}{2m-1}} \right\} \\
& \leq \int_0^\infty C e^{-\rho((\Re \mu) \cos \theta - (b_m + c' \varepsilon) \lambda^{2m}) - \rho |\operatorname{Im} \mu| \sin \theta} d\rho \int_{\mathbb{R}^n} \rho^{-n/2m} \\
& \quad \cdot \exp \left\{ -\varepsilon \varsigma_m |x|^{\frac{2m}{2m-1}} / \rho^{\frac{1}{2m-1}} \right\} dx \\
& \leq C \varepsilon^{-(n+1)} / |\mu - (b_m + c'' \varepsilon) \lambda^{2m}|
\end{aligned}$$

for all  $\Re \mu > (b_m + c'' \varepsilon) \lambda^{2m}$ ,  $0 < \varepsilon < 1$ . A scaling argument yields that

$$(2.5) \quad \|(\mu + P_\lambda(D))^{-1}\|_{L^1, L^1} \leq C \varepsilon^{-(n+1)} / |\mu - (b_m + \frac{\varepsilon}{2}) \lambda^{2m}|$$

for all  $\Re \mu > (b_m + \frac{\varepsilon}{2}) \lambda^{2m}$ ,  $0 < \varepsilon < c_1 =: 1/2c''$ .

**Step 2.** The estimates of  $(\mu + P_\lambda(D) + V)^{-1}$ . Clearly, we have

$$(\mu + P_\lambda(D) + V)^{-1} = (\mu + P_\lambda(D))^{-1} (I + V(\mu + P_\lambda(D)))^{-1}$$

and

$$V(\mu + P_\lambda(D))^{-1} = V(\mu + P(D))^{-1} [I + (P(D) - P_\lambda(D))(\mu + P_\lambda(D))^{-1}].$$

In order to proceed the proof, we claim that

$$(2.6) \quad \|(P(D) - P_\lambda(D))\varphi\|_{L^1} \leq C(\|P_\lambda(D)\varphi\|_{L^1} + (1 + \lambda^{2m})\|\varphi\|_{L^1}).$$

In fact, for any multiplex  $\alpha$  satisfying  $|\alpha| \leq 2m - 1$ , it's easy to check that  $\xi^\alpha / (1 + |\xi|^{2m})$  is an  $L^1$  multiplier. By the isometric property of multiplier under scaling, see e.g. ([21], p.145), we have

$$(2.7) \quad r^{|\alpha|} \|D^\alpha \varphi\|_{L^1} \leq C(r^{2m} \|(-\Delta)^m \varphi\|_{L^1} + \|\varphi\|_{L^1})$$

for all  $|\alpha| \leq 2m - 1$ ,  $r > 0$ ,  $\varphi \in \mathcal{L}^{2m,1}$ . Since  $P_\lambda(D)$  can be rewritten as

$$P_\lambda(D)\varphi = (-\Delta)^m \varphi + \sum_{|\beta| < |2\alpha|, q \leq |2\alpha| - |\beta|} c_{\alpha\beta} \lambda^q D^\beta \varphi,$$

for any  $\delta > 0$ , we can choose  $r = (\delta \lambda^{-q})^{\frac{1}{2m-|\beta|}}$  in (2.7) to get

$$\begin{aligned}
\|\lambda^q D^\beta \varphi\|_{L^1} & \leq C(\delta \|(-\Delta)^m \varphi\|_{L^1} + \delta^{-\frac{|\beta|}{2m-|\beta|}} \lambda^{\frac{2mq}{2m-|\beta|}} \|\varphi\|_{L^1}) \\
& \leq C(\delta \|(-\Delta)^m \varphi\|_{L^1} + \delta^{-2m+1} (1 + \lambda^{2m}) \|\varphi\|_{L^1})
\end{aligned}$$

where we have used that  $\frac{2mq}{2m-|\beta|} \leq 2m$  and  $|\beta| \leq 2m - 1$ . By the arbitrary of  $\delta$ , we obtain

$$\|P_\lambda(D)\varphi - P(D)\varphi\|_{L^1} \leq \frac{1}{2} \|P(D)\varphi\|_{L^1} + C(1 + \lambda^{2m}) \|\varphi\|_{L^1},$$

which implies (2.6).

Combining (2.5) and (2.6) yields that

$$\begin{aligned}
& \|(P(D) - P_\lambda(D))(\mu + P_\lambda(D))^{-1}\|_{L^1, L^1} \\
& \leq C(|\mu| + (1 + \lambda^{2m})) \|(\mu + P_\lambda(D))^{-1}\|_{L^1, L^1} \leq C \varepsilon^{-1}
\end{aligned}$$

for all  $\Re\mu > (b_m + \varepsilon)\lambda^{2m}$ ,  $0 < \varepsilon < c_1$ .

By the definition of  $V_\varepsilon$ , we get

$$\|V(\mu + P(D))^{-1}\|_{L^1, L^1} \leq \varepsilon^2 \|P(D)(\mu + P(D))\|_{L^1, L^1} + V_{\varepsilon^2} \|(\mu + P(D))^{-1}\|_{L^1, L^1} \leq C\varepsilon^2$$

for  $\Re\mu > \varepsilon^{-2}V_{\varepsilon^2}$ . Thus

$$\begin{aligned} & \|V(\mu + P_\lambda(D))^{-1}\|_{L^1, L^1} \\ & \leq \|V(\mu + P(D))^{-1}\|_{L^1, L^1} (1 + \|(P(D) - P_\lambda(D))(\mu + P_\lambda(D))^{-1}\|_{L^1, L^1}) \\ & \leq C\varepsilon \leq 1/2 \end{aligned}$$

for  $\Re\mu > \iota(\varepsilon, \lambda) := \varepsilon^{-2}V_{\varepsilon^2} + (b_m + \varepsilon)\lambda^{2m}$  and  $0 < \varepsilon < c_2 =: \min\{c_1, 1/2C\}$ . This implies that  $I + V(\mu + P_\lambda(D))$  is invertible, and the norm of the inverse is bounded by 2.

Therefore,

$$\|(\mu + H_{\lambda, V})^{-1}\|_{L^1, L^1} \leq C\varepsilon^{-(n+1)}/|\Im\mu|$$

for all  $\Re\mu > \iota(\varepsilon, \lambda)$ ,  $\Im\mu \neq 0$ ,  $0 < \varepsilon < c_2$ .

**Step 3.** The estimates of the semigroup. By the resolvent estimates in Step 2 and the standard method, see e.g. ([33], pp. 61–63), it can be shown that the resolvent set

$$\rho(-H_{\lambda, V}) \supset \left\{ z : |\arg(z - \iota(\varepsilon, \lambda))| < \frac{\pi}{2} + \delta \right\} \cup \{0\}$$

where  $\delta = 2^{-1} \arctan(C^{-1}\varepsilon^{n+1})$ , and in this region

$$\|(\mu + H_{\lambda, V})^{-1}\|_{L^1, L^1} \leq C\varepsilon^{-(n+1)}/|\mu - \iota(\varepsilon, \lambda)|.$$

Thus,  $-H_{\lambda, V} - \iota(\varepsilon, \lambda)$  generates an analytic semigroup in  $L^1(\mathbb{R}^n)$ , and

$$\|e^{-t(H_{\lambda, V} + \iota(\varepsilon, \lambda))}\|_{L^1, L^1} + \|t(H_{\lambda, V} + \iota(\varepsilon, \lambda))e^{-t(H_{\lambda, V} + \iota(\varepsilon, \lambda))}\|_{L^1, L^1} \leq C\varepsilon^{-(n+1)}, \quad t > 0.$$

This implies the desired conclusion obviously.  $\square$

From Theorem 2.1, we know that the operator sum  $-H_{\lambda, V}$  generates an analytic semigroup in  $L^1(\mathbb{R}^n)$ . For our purpose, to extend the semigroup in  $L^p(\mathbb{R}^n)$  ( $1 \leq p < \infty$ ), we need to show  $-H_{\lambda, V}$  also generates an analytic semigroup in  $L^2(\mathbb{R}^n)$  in some sense. Note that  $H_{\lambda, V}$  may make no sense as an operator acting on  $L^2(\mathbb{R}^n)$  for general  $V(x) \in K_{2m}(\mathbb{R}^n)$ , thus we shall work under the framework of sesquilinear forms in Hilbert spaces developed by Kato [27].

Let  $V \in K_{2m}$ , we define a sesquilinear form

$$Q_{\lambda, V}[\varphi, \psi] = (e^{-\lambda a \cdot x} (-\Delta)^{\frac{m}{2}} e^{\lambda a \cdot x} \varphi, e^{\lambda a \cdot x} (-\Delta)^{\frac{m}{2}} e^{-\lambda a \cdot x} \psi) + \int V \varphi \bar{\psi} dx,$$

where  $\varphi, \psi \in D(Q_{\lambda, V}) = H^m(\mathbb{R}^n)$ ,  $(\cdot, \cdot)$  means the inner product in  $L^2(\mathbb{R}^n)$ . Note that the form is well defined if  $V \in K_{2m}$ , see ([40], Theorem 4.2). In what follows, we denote by  $Q_{\lambda, V}[\varphi] = Q_{\lambda, V}[\varphi, \varphi]$ ,  $Q_\lambda = Q_{\lambda, 0}$ ,  $Q_0 = Q_{0, 0}$  for brevity.

**Theorem 2.2.** *The form  $Q_{\lambda, V}$  is a closed sectorial form, thus it is associated with a unique  $m$ -sectorial operator  $\tilde{H}_{\lambda, V}$  in the sense that*

$$Q_{\lambda, V}(\varphi, \psi) = (\tilde{H}_{\lambda, V} \varphi, \psi), \quad \varphi \in D(\tilde{H}_{\lambda, V}), \psi \in H^m(\mathbb{R}^n).$$

Moreover,  $-\tilde{H}_{\lambda,V}$  generates an analytic semigroup satisfying

$$\begin{aligned} \left\| e^{-t\tilde{H}_{\lambda,V}} \right\|_{L^2, L^2} + \left\| t\tilde{H}_{\lambda,V} e^{-t\tilde{H}_{\lambda,V}} \right\|_{L^2, L^2} \\ \leq C\varepsilon^{-(2m+1)} \exp \left\{ (b_m + \varepsilon)\lambda^{2m}t + C\varepsilon^{-(2m+1)}V_{\varepsilon^{4m+2}}t \right\} \end{aligned}$$

for  $\varepsilon \in (0, \varepsilon_0)$ ,  $t > 0$ .

*Proof.* In order to show that  $Q_{\lambda,V}$  is a sectorial form, we need to investigate the numerical range of  $Q_{\lambda,V}$ . To this end, for  $\varepsilon \in (0, 1)$ , it's convenient to rewrite the form  $Q_{\lambda,V}[\varphi]$  as

$$(2.8) \quad Q_{\lambda,V}[\varphi] = (1 - \varepsilon^{2m})Q_{\lambda}[\varphi] + \varepsilon^{2m}Q_{\lambda}[\varphi] + \int V|\varphi|^2.$$

In what follows we will deal with each term in the right side of (2.8) to give a lower bound of  $\Re Q_{\lambda,V}[\varphi]$ . For the first term, using (2.1), we have

$$\Re(1 - \varepsilon^{2m})Q_{\lambda}[\varphi] \geq -b_m(1 - \varepsilon^{2m})\lambda^{2m}\|\varphi\|_{L^2}^2.$$

For the second term, recall that (see [5], Lemma 7)

$$(2.9) \quad |Q_{\lambda}[\varphi] - Q_0[\varphi]| \leq \varepsilon Q_0[\varphi] + C\varepsilon^{-2m+1}(1 + \lambda^{2m})\|\varphi\|_{L^2}^2,$$

it's easy to see

$$\varepsilon^{2m}\Re Q_{\lambda}[\varphi] \geq \varepsilon^{2m}(1 - \varepsilon)Q_0[\varphi] - C\varepsilon(1 + \lambda^{2m})\|\varphi\|_{L^2}^2.$$

For the third term, we claim that

$$(2.10) \quad \left| \int V|\varphi|^2 \right| \leq \varepsilon \int |(-\Delta)^{\frac{m}{2}}\varphi|^2 + C\varepsilon^{-1}V_{\varepsilon^2} \int |\varphi|^2$$

for all  $\varphi \in H^m(\mathbb{R}^n)$ ,  $0 < \varepsilon < 1$ . In fact, let  $\nu > 0$ , similar to ([40], Theorem 2.2 and Theorem 4.2), we obtain

$$\left\| |V|^{1/2}(\nu^m + (-\Delta)^m)^{-1/2} \right\|_{L^2, L^2} \leq C \|V(\nu - \Delta)^{-m}\|_{L^1, L^1}^{1/2},$$

and

$$\|V(\nu - \Delta)^{-m}\|_{L^1, L^1} \leq C\varepsilon^2 + V_{\varepsilon^2}\nu^{-m}$$

for a constant  $C$  independent of  $\nu$ ,  $0 < \varepsilon < 1$ . Hence

$$\left\| |V|^{1/2}(\nu^m + (-\Delta)^m)^{-1/2} \right\|_{L^2, L^2} \leq C\varepsilon$$

holds for  $0 < \varepsilon < 1$  and  $\nu \geq (\frac{V_{\varepsilon^2}}{C\varepsilon^2})^{1/m}$ . This inequality gives the claim obviously.

Now replacing  $\varepsilon$  by  $\varepsilon^{2m+1}$  in (2.10), we arrive at

$$\int V|\varphi|^2 \geq -\varepsilon^{2m+1}Q_0[\varphi] - C\varepsilon^{-(2m+1)}V_{\varepsilon^{4m+2}}\|\varphi\|_{L^2}^2.$$

Combining these low bounds together implies

$$\Re Q_{\lambda,V}[\varphi] \geq -(b_m + \frac{\varepsilon}{2})\lambda^{2m}\|\varphi\|_{L^2}^2 + C\varepsilon^{2m}Q_0[\varphi] - C\varepsilon^{-(2m+1)}V_{\varepsilon^{4m+2}}\|\varphi\|_{L^2}^2,$$

for all  $\varepsilon \in (0, \varepsilon_0)$ , where  $\varepsilon_0 \in (0, 1)$  is a constant.

On the other hand, we also have

$$\begin{aligned} |\Im Q_{\lambda,V}[\varphi]| &\leq (1 + \varepsilon + \varepsilon^{2m})Q_0[\varphi] \\ &\quad + C\varepsilon^{-2m+1}(1 + \lambda^{2m})\|\varphi\|_{L^2}^2 + C\varepsilon^{-(2m+1)}V_{\varepsilon^{4m+2}}\|\varphi\|_{L^2}^2. \end{aligned}$$



Thus, there exists constants  $C', C''$  such that

$$|\Im(Q_{\lambda,V} + \iota'(\lambda, \varepsilon))[\varphi]| / |\Re(Q_{\lambda,V} + \iota'(\lambda, \varepsilon))[\varphi]| \leq C'' \varepsilon^{-2m}$$

for all  $\varphi \in H^m(\mathbb{R}^n)$ ,  $0 < \varepsilon \leq \varepsilon_0$ . Here and in the sequel,  $\iota'(\lambda, \varepsilon) = (b_m + \varepsilon)\lambda^{2m} + C'\varepsilon^{-(2m+1)}V_{\varepsilon^{4m+2}}$ .

Therefore, the numerical range of  $Q_{\lambda,V}$  contains in

$$\left\{ z \left| \left| \arg(z - \iota'(\lambda, \varepsilon)) \right| \leq \arctan(C'' \varepsilon^{-2m}) \right. \right\}.$$

Then  $Q_{\lambda,V}$  is a closed sectorial form. The existence of  $\tilde{H}_{\lambda,V}$  follows from Theorem 2.1 in (Kato [27], p. 322) directly. It suffices to prove the estimate of  $e^{-t\tilde{H}_{\lambda,V}}$  in  $L^2$ . Denote by

$$\tilde{H}_{\lambda\varepsilon,V} = \tilde{H}_{\lambda,V} + \iota'(\lambda, \varepsilon).$$

Then the numerical range  $\Theta(-\tilde{H}_{\lambda\varepsilon,V}) \subset \{z \in \mathbb{C} \mid \pi - \theta \leq |\arg z| \leq \pi\}$ ,  $\theta = \arctan(C'' \varepsilon^{-2m})$ , and the resolvent set  $\rho(-\tilde{H}_{\lambda\varepsilon,V})$  contains  $\{z \in \mathbb{C} \setminus \{0\} \mid |\arg z| < \pi - \theta\}$ . It follows from ([33], Theorem 3.9) that, for  $0 < \varepsilon \leq \varepsilon_0$ ,  $\mu \in \{z \in \mathbb{C} \setminus \{0\} \mid |\arg z| \leq \frac{1}{2}(\frac{3\pi}{2} - \theta)\}$ , we have

$$(2.11) \quad \left\| (\mu + \tilde{H}_{\lambda\varepsilon,V})^{-1} \right\|_{L^2, L^2} \leq \frac{C\varepsilon^{-2m}}{|\mu|}.$$

Therefore, for  $0 < \varepsilon \leq \varepsilon_0, t > 0$ , we have

$$\left\| e^{-t\tilde{H}_{\lambda\varepsilon,V}} \right\|_{L^2, L^2} + \left\| t\tilde{H}_{\lambda\varepsilon,V} e^{-t\tilde{H}_{\lambda\varepsilon,V}} \right\|_{L^2, L^2} \leq C\varepsilon^{-(2m+1)}.$$

This inequality gives the desired conclusion.  $\square$

We say the semigroup  $e^{-tH_{\lambda,V}}$  and  $e^{-t\tilde{H}_{\lambda,V}}$  are consistent if  $e^{-tH_{\lambda,V}}f = e^{-t\tilde{H}_{\lambda,V}}f$  is valid for any  $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ .

**Theorem 2.3.** *Semigroups  $e^{-tH_{\lambda,V}}$  on  $L^1(\mathbb{R}^n)$  and  $e^{-t\tilde{H}_{\lambda,V}}$  on  $L^2(\mathbb{R}^n)$  are consistent.*

*Proof.* For  $0 \leq l, k \leq \infty$ , set

$$V_{l,k} = \begin{cases} V(x), & -l \leq V \leq k, \\ 0, & \text{other case.} \end{cases}$$

Replacing  $V$  by  $V_{l,k}$  in the definition of  $Q_{\lambda,V}$ , we obtain a new sesquilinear form  $Q^{l,k} = Q_{\lambda,V_{l,k}}$ . Since  $V \in K_{2m}$ , it's easy to show that  $V_{l,k}$  belongs to  $K_{2m}$ ,  $Q^{l,k}$  are closed sectorial forms. Moreover,  $Q^{l,k}$  are associated with closed sectorial operators  $\tilde{H}^{l,k}$ , and  $-\tilde{H}^{l,k}$  generate analytic semigroups in  $L^2(\mathbb{R}^n)$  satisfying the same estimates with  $e^{-t\tilde{H}_{\lambda,V}}$  in Theorem 2.2.

Since  $C_0^\infty(\mathbb{R}^n)$  is a form core of  $Q_{l,k}$  and  $Q_{\infty,k}$ ,

$$\Im(Q^{l,k} - Q^{\infty,k})[\varphi] = 0, \quad \Re Q^{l,k}[\varphi] \geq \Re Q^{\infty,k}[\varphi], \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n),$$

and by Lebesgue dominated theorem

$$\lim_{l \rightarrow \infty} Q^{l,k}[\varphi] = Q^{\infty,k}[\varphi], \quad \forall \varphi \in C_0^\infty(\mathbb{R}^n),$$

an application of monotone convergence theorem from above, see ([27], p.455) implies that

$$(\mu + \tilde{H}^{l,k})^{-1} \varphi \xrightarrow[s]{} (\mu + \tilde{H}^{\infty,k})^{-1} \varphi, \quad l \rightarrow \infty$$

for all  $\varphi \in L^2(\mathbb{R}^n)$  and  $\mu > \iota'(\lambda, \varepsilon)$ .

On the other hand, by the proposition 2.1 below, we have

$$(\mu + \tilde{H}^{\infty, k})^{-1} \varphi \xrightarrow[s]{L^2} (\mu + \tilde{H}_{\lambda, V})^{-1} \varphi, \quad k \rightarrow \infty$$

for all  $\varphi \in L^2(\mathbb{R}^n)$  and  $\mu > \iota'(\lambda, \varepsilon)$ .

Thus, by Trotter approximation theorem ([27], p.504), we obtain

$$e^{-t\tilde{H}_{\lambda, V}} \varphi = \lim_{k \rightarrow \infty} \lim_{l \rightarrow \infty} e^{-t\tilde{H}^{l, k}} \varphi, \quad \text{for } t > 0, \varphi \in L^2(\mathbb{R}^n).$$

Similarly, let  $H^{l, k} = P_\lambda(D) + V_{l, k}$  with domain  $\mathcal{L}^{2m, 1}(\mathbb{R}^n)$ , then  $-H^{l, k}$  generates an analytic semigroup in  $L^1(\mathbb{R}^n)$  satisfying the same estimates in Theorem 2.1. Moreover, it follows from dominated theorem that

$$(\mu + H_{\lambda, V})^{-1} \varphi = \lim_{l, k \rightarrow \infty} (\mu + H^{l, k})^{-1} \varphi, \quad \mu > \iota(\lambda, \varepsilon), \varphi \in L^1(\mathbb{R}^n),$$

and then, by Trotter approximation theorem again, we have

$$e^{-tH_{\lambda, V}} \varphi = \lim_{l, k \rightarrow \infty} e^{-tH^{l, k}} \varphi, \quad \text{for } t > 0, \varphi \in L^1(\mathbb{R}^n).$$

The above considerations reduce the theorem to the consistency of  $e^{-t\tilde{H}^{l, k}}$  and  $e^{-tH^{l, k}}$ , and this is the case since  $V_{l, k} \in L^\infty(\mathbb{R}^n)$ , see ([16], Theorem 2).  $\square$

In the  $L^2$  part proof of Theorem 2.3, we used a convergence theorem of sectorial forms from above type to pass the limit  $l \rightarrow \infty$ . To pass the limit  $k \rightarrow \infty$ , we need a corresponding theorem from below. However, this kind of result is only available for symmetric forms in [27]. Fortunately, we can still prove the following result in our setting.

**Proposition 2.1.** *Let  $\mu > \iota'(\lambda, \varepsilon)$ ,  $\varphi \in L^2(\mathbb{R}^n)$ , then  $(\mu + \tilde{H}^{\infty, k})^{-1} \varphi \xrightarrow[s]{L^2} (\mu + \tilde{H}_{\lambda, V})^{-1} \varphi$  as  $k \rightarrow \infty$ .*

*Proof.* It's easy to check that

$$\begin{aligned} (Q^{\infty, k} + \mu)[(\mu + \tilde{H}^{\infty, k})^{-1} \varphi - (\mu + \tilde{H}_{\lambda, V})^{-1} \varphi] &+ (Q_{\lambda, V} - Q^{\infty, k})[(\mu + \tilde{H}_{\lambda, V})^{-1} \varphi] \\ &= (Q_{\lambda, V} - Q^{\infty, k})[(\mu + \tilde{H}_{\lambda, V})^{-1} \varphi, (\mu + \tilde{H}^{\infty, k})^{-1} \varphi]. \end{aligned}$$

By the property of Kato potentials, (2.9) and (2.11), we obtain

$$\begin{aligned} &\int V |(\mu + \tilde{H}^{\infty, k})^{-1} \varphi|^2 dx \\ &\leq C \left( Q^{\infty, k} [(\mu + \tilde{H}^{\infty, k})^{-1} \varphi] + (1 + \lambda^{2m}) \|(\mu + \tilde{H}^{\infty, k})^{-1} \varphi\|_{L^2}^2 \right) \\ &\leq C(\lambda, \varepsilon) \|\varphi\|_{L^2}^2 \end{aligned}$$

independent of  $k$ . Thus, by dominated convergence theorem

$$\begin{aligned}
 & \left| (Q_{\lambda,V} - Q^{\infty,k})[(\mu + \tilde{H}_{\lambda,V})^{-1}\varphi, (\mu + \tilde{H}^{\infty,k})^{-1}\varphi] \right| \\
 & \leq \left( \int |V - V_{\infty,k}| |(\mu + \tilde{H}^{\infty,k})^{-1}\varphi|^2 dx \right)^{1/2} \\
 & \quad \cdot \left( \int |V - V_{\infty,k}| |(\mu + \tilde{H}_{\lambda,V})^{-1}\varphi|^2 dx \right)^{1/2} \\
 & \leq 2C(\lambda, \varepsilon) \|\varphi\|_{L^2} \left( \int |V - V_{\infty,k}| |(\mu + \tilde{H}_{\lambda,V})^{-1}\varphi|^2 dx \right)^{1/2} \\
 & \rightarrow 0
 \end{aligned}$$

as  $k \rightarrow \infty$ . Note that  $\Re(Q^{\infty,k} + \mu) \geq \mu - \iota(\lambda, \varepsilon) > 0$  and  $Q_{\lambda,V} - Q^{\infty,k} \geq 0$ , we get

$$0 \leq (\mu - \iota(\lambda, \varepsilon)) \left[ (\mu + \tilde{H}^{\infty,k})^{-1}\varphi - (\mu + \tilde{H}_{\lambda,V})^{-1}\varphi \right] \rightarrow 0$$

as  $k \rightarrow \infty$ . This implies the desired conclusion.  $\square$

**Lemma 2.4.** *If  $V \in K_{2m}$ , then the semigroup  $e^{-tH_{\lambda,V}}$  on  $L^1(\mathbb{R}^n)$  can be extended to a strongly continuous semigroup on  $L^p(\mathbb{R}^n)$  ( $1 \leq p < \infty$ ). Moreover, it holds that*

$$\|e^{-tH_{\lambda,V}}\|_{L^1, L^\infty} \leq C\varepsilon^{-(n+1)(\lfloor \frac{n}{2m} \rfloor + 3)} t^{-\frac{n}{2m}} \exp \{t(\varepsilon^{-2}V_{\varepsilon^2} + b_m(1 + \varepsilon)\lambda^{2m})\}$$

where  $[s]$  denotes the integer part of real number  $s$ ,  $t > 0, 0 < \varepsilon < c$ .

*Proof.* The first statement follows from Theorem 2.1, 2.2, 2.3 and duality, interpolation arguments, thus it suffices to prove the desired estimates. Since

$$e^{-H_{\lambda,V}} : L^1(\mathbb{R}^n) \mapsto \mathcal{L}^{2m,1}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$$

for  $1 < p < \frac{n}{n-2m}$  if  $2m < n$  and  $1 < p < \infty$  if  $2m \geq n$ . Especially, we can choose  $\frac{1}{p} = 1 - \frac{1}{N}$ ,  $N = \lfloor \frac{n}{2m} \rfloor + 2$ . Then combining the argument in ([35], p.463) and Theorem 2.1 yields that

$$(2.12) \quad \|e^{-H_{\lambda,V}}\|_{L^1, L^\infty} \leq C\varepsilon^{-(n+1)N} \exp \{ \varepsilon^{-2}V_{\varepsilon^2} + b_m(1 + \varepsilon)\lambda^{2m} \}, \quad 0 < \varepsilon < c.$$

Now we are in place to get the estimate of  $\|e^{-tH_{\lambda,V}}\|_{L^1, L^\infty}$  for  $t > 0$ . Following [40], we use a scaling argument. Let  $H_{\lambda,V,t} = e^{-\lambda a \cdot (t^{\frac{1}{2m}} \cdot)} (-\Delta)^m e^{\lambda a \cdot (t^{\frac{1}{2m}} \cdot)} + tV(t^{\frac{1}{2m}} \cdot)$ ,  $U_t g = g(t^{\frac{1}{2m}} \cdot)$ ,  $0 < t < 1$ . Then

$$tH_{\lambda,V}\varphi = U_t^{-1}H_{\lambda,V,t}U_t\varphi, \quad \text{for } \varphi \in \mathcal{L}^{2m,1}(\mathbb{R}^n)$$

and

$$e^{-t\tau H_{\lambda,V}}\varphi = U_t^{-1}e^{-\tau H_{\lambda,V,t}}U_t\varphi, \quad \text{for } \varphi \in L^1(\mathbb{R}^n).$$

Similar to (2.12), we have for  $0 < t < 1, 0 < \varepsilon < c$

$$\|e^{-H_{\lambda,V,t}}\|_{L^1, L^\infty} \leq C\varepsilon^{-(n+1)N} \exp \{t(\varepsilon^{-2}V_{\varepsilon^2} + b_m(1 + \varepsilon)\lambda^{2m})\}.$$

Thus, for  $0 < t < 1$  we obtain

$$\begin{aligned}
 \|e^{-tH_{\lambda,V}}\|_{L^1, L^\infty} & \leq \|U_t^{-1}\|_{L^\infty, L^\infty} \|e^{-H_{\lambda,V,t}}\|_{L^1, L^\infty} \|U_t\|_{L^1, L^1} \\
 & \leq C\varepsilon^{-(n+1)N} t^{-\frac{n}{2m}} \exp \{t(\varepsilon^{-2}V_{\varepsilon^2} + b_m(1 + \varepsilon)\lambda^{2m})\}.
 \end{aligned}$$

For  $t > 1$ , it follow from Theorem 2.1 and (2.12) that

$$\begin{aligned} \|e^{-tH_{\lambda,V}}\|_{L^1,L^\infty} &\leq \|e^{-H_{\lambda,V}}\|_{L^1,L^\infty} \|e^{-(t-1)H_{\lambda,V}}\|_{L^1,L^1} \\ &\leq C\varepsilon^{-(n+1)(N+1)} \exp\{\varepsilon^{-2}V_{\varepsilon^2} + b_m(1+\varepsilon)\lambda^{2m}\}. \end{aligned}$$

The two estimates imply the desired conclusion.  $\square$

We can now finish the proof of Theorem 1.3.

*The proof of Theorem 1.3.* From Lemma 2.4 we know that  $e^{-t((-\Delta)^m+V)}$  is bounded from  $L^1(\mathbb{R}^n)$  to  $L^\infty(\mathbb{R}^n)$ , then ([35], Corollary A.1.2) there exists a measurable function  $K$  on  $\mathbb{R}^n \times \mathbb{R}^n$  such that

$$(e^{-t((-\Delta)^m+V)}\varphi)(x) = \int K(t,x,y)\varphi(y)dy, \quad \text{for } \varphi \in L^1(\mathbb{R}^n).$$

Since  $e^{-tH_{\lambda,V}}$  has the kernel

$$K_{\lambda,a}(t,x,y) = e^{-\lambda a \cdot x} K(t,x,y) e^{\lambda a \cdot y},$$

which satisfies the same bound as  $\|e^{-tH_{\lambda,V}}\|_{L^1,L^\infty}$ , we have

$$\begin{aligned} |K(t,x,y)| &\leq C\varepsilon^{-(n+1)([\frac{n}{2m}]+3)} t^{-\frac{n}{2m}} \\ &\quad \cdot \exp\{\lambda a \cdot x - \lambda a \cdot y + t(\varepsilon^{-2}V_{\varepsilon^2} + b_m(1+\varepsilon)\lambda^{2m})\}. \end{aligned}$$

Optimizing the above estimate with respect to  $a$  and then  $\lambda > 0$  yields Theorem 1.3.

*Remark 2.5.* The problem that whether or not Theorem 1.3 is valid for  $V \in K_{2m}$  with  $m \geq 1$  was originally raised by the third author after the joint work [40] with Yao, where they proved the weaker result that  $\|e^{-tH}\|_{L^1,L^\infty} \leq Ct^{-\frac{n}{2m}}$ .

### 3. SHARP HEAT KERNEL ESTIMATES FOR THE FRACTIONAL CASE

In this section, we shall prove Theorem 1.4, unless stated otherwise, here and throughout this section,  $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$ . We use the notations  $a \wedge b = \min\{a, b\}$ ,  $a \vee b = \max\{a, b\}$  for  $a, b \in \mathbb{R}$ .

First of all, let us recall the kernel estimates in the free case, indeed, its asymptotic behavior can be found in [7].

**Lemma 3.1.** *Let  $K_0(t,x) = \mathcal{F}^{-1}(e^{-t|\cdot|^{2\alpha}})(x)$  and  $I(t,x) = \frac{t}{|x|^{n+2\alpha}} \wedge t^{-\frac{n}{2\alpha}}$  for  $t > 0$  and  $x \in \mathbb{R}^n$ , then  $K_0$  is a smooth function for  $t > 0$  and satisfies*

$$(3.1) \quad |K_0(t,x)| \leq C_1 I(t,x).$$

Before proceeding, we note that there exists some constant  $C$  such that

$$\frac{t}{(|x|^2 + t^{\frac{1}{\alpha}})^{\frac{n}{2} + \alpha}} \leq I(t,x) \leq C \frac{t}{(|x|^2 + t^{\frac{1}{\alpha}})^{\frac{n}{2} + \alpha}}.$$

Next, we need the following characterization of Kato potentials, where the case  $0 < 2\alpha < n$  is essentially due to Bogdan and Jackubowski [8].

**Lemma 3.2.**  *$V \in K_{2\alpha}(\mathbb{R}^n)$  if and only if  $\lim_{t \rightarrow 0} K_V(t) = 0$ , where*

$$K_V(t) = \sup_x \int_{\mathbb{R}^n} J(t,x-y)|V(y)|dy,$$

and

$$J(t, x) = \begin{cases} |x|^{2\alpha-n} \wedge t^2|x|^{-n-2\alpha}, & 0 < 2\alpha < n, \\ (1 \vee \ln(t|x|^{-n})) \wedge t^2|x|^{-2n}, & 2\alpha = n, \\ t^{1-\frac{n}{2\alpha}} \wedge t^2|x|^{-n-2\alpha}, & 2\alpha > n. \end{cases}$$

*Proof.* We first estimate the resolvent kernel of  $(-\Delta)^\alpha$  by using Lemma 3.1. If we denote

$$R(\mu, x) = \int_0^\infty e^{-t\mu} K_0(t, x) dt, \quad \mu > 0,$$

which is the Laplace transform of the kernel of  $e^{-t(-\Delta)^\alpha}$ , then it's not hard to verify that  $R(\mu, \cdot) \in L^1(\mathbb{R}^n)$  and  $\mathcal{F}^{-1}R(\mu, \cdot)(\xi) = 1/(\mu + |\xi|^{2\alpha})$ , hence  $R(\mu, x - y)$  is the integral kernel of  $(\mu + (-\Delta)^\alpha)^{-1}$ . We claim that

$$|R(\mu, x)| \leq C_2 J(\mu^{-1}, x) \quad \text{for } x \in \mathbb{R}^n \setminus \{0\} \text{ and } \mu > 0,$$

In fact, the case  $2\alpha \neq n$  follows from the same proof in Lemma 7 of [8]. If  $2\alpha = n$ , then by scaling, it suffices to show

$$\int_0^\infty e^{-t} I(t, x) dt \leq C_2 (1 \vee |\ln |x|^n|) \wedge |x|^{-2n}.$$

A direct computation shows that

$$\int_0^{|x|^n} t e^{-t} dt = 1 - e^{-|x|^n} - |x|^n e^{-|x|^n} \leq 1 \wedge |x|^{2n}.$$

Also, we have both

$$\int_{|x|^n}^\infty t^{-1} e^{-t} dt \leq |x|^{-n} e^{-|x|^n} \leq |x|^{-2n}$$

and

$$\int_{|x|^n}^\infty t^{-1} e^{-t} dt \leq 1 + \left| \int_{|x|^n}^1 t^{-1} dt \right| = 1 + |\ln |x|^n|.$$

Combine these together we prove the case  $2\alpha = n$ . Note that

$$\|(\mu + (-\Delta)^\alpha)^{-1} V\|_{L^\infty} = \sup_x \left| \int R_\mu(x - y) |V(y)| dy \right| \leq C_2 K_V(\mu^{-1})$$

and recall that (see [40])  $V \in K_{2\alpha}(\mathbb{R}^n)$  if and only if

$$\lim_{\mu \rightarrow \infty} \|(\mu + (-\Delta)^\alpha)^{-1} V\|_{L^\infty} = 0.$$

Therefore,  $\lim_{t \rightarrow 0} K_V(t) = 0$  implies that  $V \in K_{2\alpha}(\mathbb{R}^n)$ .

Conversely, the case  $0 < 2\alpha < n$  follows from the same argument of Corollary 12 in [8]. The idea to prove other cases is similar. If  $V \in K_n$  and  $t \in (0, e^{-1})$ , then

$$\begin{aligned} & \int_{\mathbb{R}^n} ((1 \vee \ln \frac{t}{|x-y|^n}) \wedge \frac{t^2}{|x-y|^{2n}}) |V(y)| dy \\ &= \int_{|x-y|^n < t} (1 \vee \ln \frac{t}{|x-y|^n}) |V(y)| dy + \int_{|x-y|^n \geq t} \frac{|V(y)|}{|x-y|^{2n}} t^2 dy \\ &\leq 2 \int_{|x-y|^n < t} \ln |x-y|^{-n} |V(y)| dy + \int_{|x-y|^n \geq t} \frac{|V(y)|}{|x-y|^{2n}} t^2 dy. \end{aligned}$$

The first term goes to 0 as  $t \rightarrow 0$  (uniformly for  $x \in \mathbb{R}^n$ ) by the definition of Kato class. To deal with the second term, we denote  $\mathcal{C}_{x,r} = \{y \in \mathbb{R}^n : |y_i - x_i| < r\}$  for  $x \in \mathbb{R}^n$ ,  $r > 0$ , and  $K_s = \{a \in \mathbb{Z}^n : \max |a_i| = s\}$  for  $s \in \mathbb{N}$ . Put  $r = t^{\frac{1}{n}}$ , we derive

$$\begin{aligned} \int_{|x-y|^n \geq t} \frac{|V(y)|}{|x-y|^{2n}} t^2 dy &\leq \sum_{a \in \mathbb{Z}^n \setminus \{0\}} \int_{\mathcal{C}_{x+ar, r/2}} \frac{|V(y)|}{|x-y|^{2n}} r^{2n} dy \\ &= \sum_{s=1}^{\infty} \sum_{a \in K_s} \int_{\mathcal{C}_{x+ar, r/2}} \frac{|V(y)|}{|x-y|^{2n}} r^{2n} dy \\ &\leq \sum_{s=1}^{\infty} \sum_{a \in K_s} \left(s - \frac{1}{2}\right)^{-2n} \int_{\mathcal{C}_{x+ar, r/2}} |V(y)| dy \\ &\leq C_3 \int_{\mathcal{C}_{x+ar, r/2}} |V(y)| dy, \end{aligned}$$

where we notice that  $\sum_{a \in K_s} 1 = n(s+1)^{n-1}$ . Since  $V \in K_n$  implies that

$$\lim_{r \rightarrow 0} \sup_x \int_{\mathcal{C}_{x+ar, r/2}} |V(y)| dy = 0,$$

we prove the result in the case  $2\alpha = n$ .

Finally, if  $V \in K_{2\alpha}(\mathbb{R}^n)$  with  $2\alpha > n$ , then  $\sup_x \int_{|x-y| < 1} |V(y)| dy < \infty$ , and thus by the same argument as above we find

$$\sup_x \int_{\mathbb{R}^n} (1 \wedge \frac{t^{\frac{n}{2\alpha}+1}}{|x-y|^{n+2\alpha}}) |V(y)| dy < \infty,$$

which completes the proof.  $\square$

Like in section 2, we define a number which will appear in the kernel estimates below. For  $0 < \varepsilon \ll 1$ , and  $\omega = eC_1C_2C_4$ ,  $C_4 = 2^{\alpha-1} \vee 2^{\frac{n}{2\alpha}}$ , we set

$$V^\varepsilon = \sup \{\sigma : \sigma \in F_\varepsilon\},$$

where

$$F_\varepsilon = \left\{ \sigma \leq 1 : t \in (0, \sigma), \omega K_V(t) \leq \varepsilon \right\}.$$

We are now ready to prove Theorem 1.4.

*The proof of Theorem 1.4.* Let

$$(3.2) \quad K_j(t, x, y) = \int_{\mathbb{R}^n} \int_0^t K_{j-1}(t-s, x, z) V(z) K_0(s, z, y) ds dz \quad \text{for } j \in \mathbb{N},$$

where  $K_0(t, x, y) = K_0(t, x-y)$ .

**Step 1.** We will prove by induction that

$$(3.3) \quad |K_j(t, x, y)| \leq C_1 (\omega K_V(t))^j I(t, x-y) \quad \text{for } j \in \mathbb{N} \cup \{0\}.$$

If  $j = 0$ , then (3.3) follows immediately from Lemma 3.1. We first notice that

$$\begin{aligned} I(t, x) \wedge I(s, y) &\leq (2^{\alpha-1} \frac{t+s}{|x+y|^{n+2\alpha}}) \wedge (2^{\frac{n}{2\alpha}} (t+s)^{-\frac{n}{2\alpha}}) \\ &\leq C_4 I(t+s, x+y), \end{aligned}$$

and thus

$$(3.4) \quad \begin{aligned} I(t, x)I(s, y) &= (I(t, x) \wedge I(s, y))(I(t, x) \vee I(s, y)) \\ &\leq C_4 I(t + s, x + y)(I(t, x) \vee I(s, y)). \end{aligned}$$

Meanwhile, by the proof of Lemma 3.2 we see that

$$\int_0^t I(t - s, x) ds = \int_0^t I(s, x) ds \leq e \int_0^\infty e^{-s/t} I(s, x) ds \leq e C_2 J(t, x).$$

Since for every  $x \in \mathbb{R}^n$ ,  $t \mapsto J(t, x)$  is a nondecreasing function, it follows by (3.2), inductive assumption and the definition of  $K_V(t)$  that

$$\begin{aligned} |K_j(t, x, y)| &\leq C_1^2 (\omega K_V(t))^{j-1} \int_{\mathbb{R}^n} \int_0^t (I(t - s, x - z) I(s, z - y)) |V(z)| ds dz \\ &\leq C_1^2 C_4 (\omega K_V(t))^{j-1} I(t, x - y) \int_{\mathbb{R}^n} \int_0^t (I(t - s, x - z) \vee I(s, z - y)) |V(z)| ds dz \\ &\leq e C_1^2 C_2 C_4 (\omega K_V(t))^{j-1} I(t, x - y) \int_{\mathbb{R}^n} (J(t, x - z) \vee J(t, z - y)) |V(z)| dz \\ &\leq C_1 (\omega K_V(t))^j I(t, x - y). \end{aligned}$$

Therefore the proof of (3.3) is completed.

**Step 2.** We will show that

$$(3.5) \quad \lim_{N \rightarrow \infty} \|e^{-tH} - \sum_{j=0}^N (-1)^j T_j(t)\|_{L^1, L^1} = 0 \quad \text{for } t > 0,$$

where

$$(T_j(t)f)(x) = \int_{\mathbb{R}^n} K_j(t, x, y) f(y) dy \quad \text{for } f \in L^1(\mathbb{R}^n).$$

We note that  $V \in K_{2\alpha}(\mathbb{R}^n)$  implies that  $\sup_x \int_{|x-y|<1} |V(y)| dy \leq C$ , then by (3.3), the definition of  $K_V(t)$  and the fact that  $\int I(t, x) dx = C$ , which is independent of  $t$ , we see that  $T_j(t) : L^1(\mathbb{R}^n) \rightarrow L^1(\mathbb{R}^n)$  is bounded with polynomial growth of large  $t > 0$  for  $j \in \mathbb{N} \cup \{0\}$ .

We recall that  $e^{-z(-\Delta)^\alpha}$  is an analytic semigroup of angle  $\frac{\pi}{2}$  on  $L^1(\mathbb{R}^n)$  (see [29]) and that  $V$  is a Kato perturbation of  $(-\Delta)^\alpha$  in  $L^1(\mathbb{R}^n)$  (see [40]), that is, for each  $\varepsilon > 0$  there exists  $C_\varepsilon > 0$  such that

$$\|V\varphi\|_{L^1} \leq \varepsilon \|(-\Delta)^\alpha \varphi\|_{L^1} + C_\varepsilon \|\varphi\|_{L^1} \quad \text{for } \varphi \in \mathcal{L}^{2\alpha, 1}(\mathbb{R}^n).$$

Thus, for fixed  $\theta_0 \in (0, \frac{\pi}{2})$  and sufficient large  $\omega_0 > 0$ , choosing small  $\varepsilon$  yields that

$$\begin{aligned} \|V(\mu + (-\Delta)^\alpha)^{-1}\|_{L^1, L^1} &\leq \varepsilon \|(-\Delta)^\alpha (\mu + (-\Delta)^\alpha)^{-1}\|_{L^1, L^1} \\ &\quad + C_\varepsilon \|(\mu + (-\Delta)^\alpha)^{-1}\|_{L^1, L^1} \\ &< \frac{1}{2} \quad \text{for } \mu \in \omega_0 + \Sigma_{\theta_0 + \frac{\pi}{2}}. \end{aligned}$$

Consequently, we have

$$(\mu + H)^{-1} = \sum_{j=0}^{\infty} (-1)^j (\mu + (-\Delta)^\alpha)^{-1} (V(\mu + (-\Delta)^\alpha)^{-1})^j$$

for such  $\mu$ , in particular,

$$\sup\{\|V(\mu + H)^{-1}\|_{L^1, L^1} : \mu \in \omega_0 + \Sigma_{\theta_0 + \frac{\pi}{2}}\} \leq 2.$$

Combining these statements we find that

$$r_N(\mu) = (\mu + (-\Delta)^\alpha)^{-1} (V(\mu + (-\Delta)^\alpha)^{-1})^N V(\mu + H)^{-1}, \quad \mu \in \omega_0 + \Sigma_{\theta_0 + \frac{\pi}{2}},$$

is an analytic function satisfying that

$$\sup\{\|(\mu - \omega_0)r_N(\mu)\|_{L^1, L^1} : \mu \in \omega_0 + \Sigma_{\theta_0 + \frac{\pi}{2}}\} \leq C_5 2^{-N}.$$

It follows now from Theorem 2.6.1 in [1] that

$$\left\| \int_{\Gamma} e^{\mu t} r_N(\mu) d\mu \right\|_{L^1, L^1} \leq C_6 2^{-N} e^{\omega_0 t} \rightarrow 0 \quad (N \rightarrow \infty),$$

where  $\Gamma = \Gamma_0 \cup \Gamma_{\pm}$ ,  $\Gamma_0 = \{\mu : \mu = \omega_0 + \delta e^{i\theta}, |\theta| \leq \theta_1 + \frac{\pi}{2}\}$  and  $\Gamma_{\pm} = \{\mu : \mu = \omega_0 + r e^{\pm i(\theta_1 + \frac{\pi}{2})}, r \geq \delta\}$  ( $0 < \theta_1 < \theta_0$ ,  $\delta > 0$ ). Similarly, for every  $j \in \mathbb{N} \cup \{0\}$  the integral

$$\int_{\Gamma} e^{zt} (\mu + (-\Delta)^\alpha)^{-1} (V(\mu + (-\Delta)^\alpha)^{-1})^j d\mu$$

exists and its Laplace transform is  $(\mu + (-\Delta)^\alpha)^{-1} (V(\mu + (-\Delta)^\alpha)^{-1})^j$ . Thanks to the following finite expansion

$$(\mu + H)^{-1} - \sum_{j=0}^N (-1)^j (\mu + (-\Delta)^\alpha)^{-1} (V(\mu + (-\Delta)^\alpha)^{-1})^j = (-1)^{N+1} r_N(\mu),$$

we obtain that

$$\sum_{j=0}^N (-1)^j \int_{\Gamma} e^{zt} (\mu + (-\Delta)^\alpha)^{-1} (V(\mu + (-\Delta)^\alpha)^{-1})^j d\mu \rightarrow \int_{\Gamma} e^{\mu t} (\mu + H)^{-1} d\mu$$

in operator norm on  $L^1(\mathbb{R}^n)$  as  $N$  goes to infinity. Since  $e^{-tH} = \int_{\Gamma} e^{\mu t} (\mu + H)^{-1} d\mu$  follows from the representation of analytic semigroups, it suffices by the uniqueness of Laplace transforms to show that the Laplace transform of each  $T_j(t)$  is  $(\mu + (-\Delta)^\alpha)^{-1} (V(\mu + (-\Delta)^\alpha)^{-1})^j$ .

To this end, we first note that  $K_j(t, x, y)$  is jointly continuous when  $t > 0$  by using (3.2), Lemma 3.2 and dominated convergence theorem (see Lemma 14 in [8]), then by (3.3) and dominated convergence theorem again, we see that  $T_j(t)$  ( $t > 0$ ) is strongly continuous on  $L^1$ .

Next, we denote

$$R_j(\mu, x, y) = \int R_{j-1}(\mu, x, z) V(z) R_0(\mu, z, y) dz, \quad j \geq 1,$$

where  $R_0(\mu, x, y) = R(\mu, x - y)$ . According to Lemma 3.2 and the fact that

$$J(\mu^{-1}, x - z) \wedge J(\mu^{-1}, z - y) \leq C J(\mu^{-1}, x - y),$$

we have the following

$$(3.6) \quad |R_j(\mu, x, y)| \leq C K_V(\mu^{-1})^j J(\mu^{-1}, x - y)$$

Hence,  $R_j(\mu, x, y)$  is well defined for each  $j$ , and is actually the kernel of the operator  $(\mu + (-\Delta)^\alpha)^{-1} (V(\mu + (-\Delta)^\alpha)^{-1})^j$ . Now we will prove by an induction argument to show that the Laplace transform of  $K_j(t, x, y)$  is  $R_j(\mu, x, y)$ . The case  $j = 0$  follows by definition. Similar to the proof of estimate (3.6), We derive

$$\int_0^\infty e^{-t\mu} |K_j(t, x, y)| dt \leq C K_V(\mu^{-1})^j J(\mu^{-1}, x - y),$$



then applying Fubini's theorem and induction assumption, we have

$$\begin{aligned} \int_0^\infty e^{-t\mu} K_{j+1}(t, x, y) dt &= \int_0^\infty e^{-t\mu} \int_{\mathbb{R}^n} \int_0^t K_j(t-t_1, x, z) V(z) K_0(t_1, z, y) dt_1 dz \\ &= \int_{\mathbb{R}^n} V(z) \int_0^\infty e^{-t\mu} K_j(t, x, z) dt \int_0^\infty e^{-t_1\mu} K_0(t_1, z, y) dt_1 dz \\ &= \int_{\mathbb{R}^n} R_j(\mu, x, z) V(z) R(\mu, z, y) dz \\ &= R_{j+1}(\mu, x, y). \end{aligned}$$

Finally, a direct computation shows

$$\int_{\mathbb{R}^n} \int_0^\infty |e^{-t\mu} K_j(t, x, y)| dt dy \leq CK_V(\mu^{-1})^j \int_{\mathbb{R}^n} J(\mu^{-1}, x-y) dy \leq \frac{CK_V(\mu^{-1})^j}{\mu},$$

which implies again by Fubini's theorem that

$$\begin{aligned} \int_0^\infty e^{-t\mu} T_j(t) f(x) dt &= \int_0^\infty e^{-t\mu} \int_{\mathbb{R}^n} K_j(t, x, y) f(y) dy dt \\ &= \int_{\mathbb{R}^n} R_\mu^j(x, y) f(y) dy \\ &= (\mu + (-\Delta)^\alpha)^{-1} (V(\mu + (-\Delta)^\alpha)^{-1})^j f(x). \end{aligned}$$

**Step 3.** Now, for given  $\varepsilon \in (0, 1)$ , and  $t \in (0, V^\varepsilon)$ , if we define

$$K(t, x, y) = \sum_{j=0}^\infty K_j(t, x, y),$$

and the associated operator

$$T(t)f(x) = \int K(t, x, y) f(y) dy, \quad f \in L^1.$$

then it follows that

$$|K(t, x, y)| \leq \sum_{j=0}^\infty C_1(\omega K_V(t))^j I(t, x-y) \leq \frac{C_1}{1-\varepsilon} I(t, x-y).$$

and

$$\lim_{N \rightarrow \infty} \|T(t) - \sum_{j=0}^N (-1)^j T_j(t)\|_{L^1, L^1} = 0 \quad \text{for } 0 < t < V^\varepsilon.$$

In view of (3.5), we have proved that  $K(t, x, y)$  coincides with the kernel of  $e^{-tH}$  for  $t$  small enough. For convenience, we still denote  $K(t, x, y)$  the kernel of  $e^{-tH}$  for any  $t > 0$ , and by semigroup property, we can pass the estimates above to the general case. Indeed, when  $t \in (V^\varepsilon, 2V^\varepsilon)$ , since  $K(t, x, y) = \int_{\mathbb{R}^n} K(t/2, x, z) K(t/2, z, y) dz$ , and  $\int_{\mathbb{R}^n} I(t; x-y) dy = C_8$  is independent of  $t$  and  $x$ , it follows from (3.4) that

$$\begin{aligned} |K(t, x, y)| &\leq \left(\frac{C_1}{1-\varepsilon}\right)^2 I(t, x-y) \int_{\mathbb{R}^n} |K(t/2, x, z)| + |K(t/2, z, y)| dz \\ &\leq 2C_4 C_8 \left(\frac{C_1}{1-\varepsilon}\right)^2 I(t, x-y) \end{aligned}$$

By doing this inductively, we have for  $t \in (2^{n-1}V^\varepsilon, 2^nV^\varepsilon)$ ,

$$|K(t, x, y)| \leq \frac{1}{2C_4C_8} \left( \frac{2C_1C_4C_8}{1-\varepsilon} \right) 2^n I(t, x-y).$$

If we choose  $\mu_{\varepsilon, V} = \frac{C}{V^\varepsilon}$  with some constant  $C$ , then we obtain that

$$|K(t, x, y)| \leq Ce^{\mu_{\varepsilon, V} t} I(t, x-y),$$

which completes the proof of Theorem 1.4.

*Remark 3.3.* We would like to point out that property (3.4) plays an important role in our proof, however, it fails for function like

$$t^{-\frac{n}{2m}} \exp\left\{-\varsigma_m \frac{|x-y|^{\frac{2m}{2m-1}}}{t^{\frac{1}{2m-1}}}\right\},$$

which is the associated upper bound of the kernel of the semigroup  $e^{-t(-\Delta)^m}$  for positive integer  $m$  appeared in section 2, so it seems that the method here we treat for the fractional case does not apply to the case we have done in section 2.

#### 4. $L^p$ ESTIMATES FOR FRACTIONAL SCHRÖDINGER EQUATION

As already mentioned in the introduction, Theorem 1.2 can be easily achieved from Theorem 1.3 (see [11]) since in that case, we have the following stronger estimates

$$(4.1) \quad \|e^{-zH}\|_{L^p, L^p} \leq C \left( \frac{|z|}{|\Re z|} \right)^{n_p + \varepsilon}, \quad 1 \leq p \leq \infty$$

valid for all  $\Re z > 0$  and  $\varepsilon > 0$ . However, we don't know whether or not estimates (4.1) are still true in the fractional case.

Instead, the goal of this section is to prove Theorem 1.2 by adapting the method in [25, 26]. First, we recall the amalgams of  $L^q$  and  $l^p$  consisting function such that

$$l^p(L^q) = \{\varphi \in L^q_{loc} : \sum_k \|\varphi\|_{L^q(\mathcal{C}(k))}^p < \infty\},$$

where  $\mathcal{C}(k)$  is the unit cube centered at  $k$ ,  $k \in \mathbb{Z}^n$ . More facts on  $l^p(L^q)$ -spaces can be found in [20].

**Theorem 4.1.** *Let  $1 \leq p \leq q \leq \infty$ ,  $\lambda > \frac{n}{2\alpha}(\frac{1}{p} - \frac{1}{q})$ , and  $H_\theta = (-\Delta)^\alpha + \theta V(\theta^{\frac{1}{2\alpha}} \cdot)$ , then for sufficiently large constant  $M$ ,  $(H_\theta + M)^{-\lambda}$  is uniformly bounded from  $L^p(\mathbb{R}^n)$  to  $l^p(L^q)$  with respect to  $\theta \in (0, 1]$ .*

*Proof.* Let  $K_\theta(t, x, y)$  be the integral kernel of  $e^{-tH_\theta}$ , and denote  $\tilde{U}_\theta g = \theta^{\frac{n}{2\alpha}} g(\theta^{\frac{1}{2\alpha}} \cdot)$ , we have

$$e^{-\theta H_1} g = \tilde{U}_\theta^{-1} e^{-H_\theta} \tilde{U}_\theta g, \quad g \in L^1,$$

thus  $K_\theta(t, x, y) = \theta^{\frac{n}{2\alpha}} K_1(\theta t, \theta^{\frac{1}{2\alpha}} x, \theta^{\frac{1}{2\alpha}} y)$ . We claim that

$$(4.2) \quad \|K_\theta(t, \cdot)\|_{l^1(L^p)} \leq Ce^{Lt} (1 + t^{-\frac{n}{2\alpha}(1-\frac{1}{p})}), \quad t > 0.$$

When  $\alpha = m$  is an integer, it follows from Theorem 1.3 with some given  $\varepsilon > 0$  and constant  $c_m$  that

$$\begin{aligned} |K_\theta(t, x, y)| &\leq C\theta^{\frac{n}{2m}}(\theta t)^{-\frac{n}{2m}}e^{L\theta t}\exp\left(-c_m\frac{|\theta^{\frac{1}{2m}}x - \theta^{\frac{1}{2m}}y|^{\frac{2m}{2m-1}}}{(\theta t)^{\frac{1}{2m-1}}}\right) \\ &\leq Ct^{-\frac{n}{2m}}e^{Lt}\exp\left(-c_m\frac{|x-y|^{\frac{2m}{2m-1}}}{t^{\frac{1}{2m-1}}}\right), \end{aligned}$$

where  $C, L, c_m$  are independent of  $\theta$ . We can exploit this upper bound to estimate the  $l^1(L^p)$  norm of  $K_\theta(t, x, y)$ . Indeed, combine

$$\begin{aligned} \|K_\theta(t, \cdot)\|_{L^p(\mathcal{C}(0))} &\leq Ct^{-\frac{n}{2m}}e^{Lt}\left(\int \exp\left(-pc_m\frac{|x|^{\frac{2m}{2m-1}}}{t^{\frac{1}{2m-1}}}\right)dx\right)^{\frac{1}{p}} \\ &\leq Ct^{-\frac{n}{2m}(1-\frac{1}{p})}e^{Lt}, \end{aligned}$$

and

$$\begin{aligned} \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \|K_\theta(t, \cdot)\|_{L^p(\mathcal{C}(k))} &\leq Ct^{-\frac{n}{2m}}e^{Lt} \sum_{k \in \mathbb{Z}^n \setminus \{0\}} e^{-c|k|^{\frac{2m}{2m-1}}/t^{\frac{1}{2m-1}}} \\ &\leq Ce^{Lt}, \end{aligned}$$

we derive estimate (4.2) for integer  $\alpha$ .

When  $\alpha$  is not an integer, it now follows from Theorem 1.4 that

$$|K_\theta(t, x, y)| \leq Ce^{\mu_\varepsilon, \nu^t} I(t, x - y).$$

It's easy to see that

$$\sum_k \|I(t, \cdot)\|_{L^p(\mathcal{C}(k))} \leq I_1 + I_2,$$

where

$$\begin{aligned} I_1 &= \sum_k \left( \int_{\mathcal{C}(k) \cap \{|x| \leq t^{\frac{1}{2\alpha}}\}} I(t, x)^p dx \right)^{\frac{1}{2}}, \\ I_2 &= \sum_k \left( \int_{\mathcal{C}(k) \cap \{|x| > t^{\frac{1}{2\alpha}}\}} I(t, x)^p dx \right)^{\frac{1}{2}}. \end{aligned}$$

Since

$$I_1 \leq \left( \int_{\mathbb{R}^n} I(t, x)^p dx \right)^{\frac{1}{p}} \leq Ct^{-\frac{n}{2\alpha}(1-\frac{1}{p})},$$

and

$$I_2 \leq t \sum_{k \neq 0} \sup_{x \in \mathcal{C}(k) \cap \{|x| \geq t^{\frac{1}{2\alpha}}\}} \frac{1}{|x|^{n+2\alpha}} \leq t \int_{t^{\frac{1}{2\alpha}}}^{\infty} \frac{dx}{x^{1+2\alpha}} = C.$$

Thus, we also obtain (4.2).

Using estimate (4.2) with the young-type inequality of  $l^p(L^q)$  (see [26]), we have

$$\begin{aligned} \|e^{-tH_\theta} \varphi\|_{l^p(L^q)} &\leq \|K_\theta(t, \cdot) * \varphi\|_{l^p(L^q)} \\ &\leq C \|K_\theta(t, \cdot)\|_{l^1(L^r)} \|\varphi\|_{L^p} \\ &\leq Ce^{Lt} (1 + t^{-\frac{n}{2\alpha}(\frac{1}{p} - \frac{1}{q})}) \|\varphi\|_{L^p}, \end{aligned}$$

where  $\varphi \in L^p$ ,  $\frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{q}$ . Therefore, according to the formula

$$(H_\theta + M)^{-\lambda} = \frac{1}{\Gamma(\lambda)} \int_0^\infty t^{\lambda-1} e^{-Mt} e^{-tH_\theta} dt,$$

we get

$$\|(H_\theta + M)^{-\lambda} \varphi\|_{L^p(L^q)} \leq \frac{1}{\Gamma(\lambda)} \|\varphi\|_{L^p} \int_0^\infty (t^{\lambda - \frac{n}{2\alpha}(\frac{1}{p} - \frac{1}{q}) - 1} + t^{\lambda-1}) e^{-(M-L)t} dt.$$

Since  $\lambda > \frac{n}{2\alpha}(\frac{1}{p} - \frac{1}{q})$ , the integral in the right hand side is finite if  $M > L$ .  $\square$

**Theorem 4.2.** *Let  $\beta > 0$ ,  $R_\theta = (H_\theta + M)^{-1}$ , then there is a positive constant  $C$  which is independent of  $\theta \in (0, 1]$  and  $k \in \mathbb{Z}^n$  such that*

$$\|\langle \cdot - k \rangle^\beta e^{itR_\theta} \langle \cdot - k \rangle^{-\beta}\|_{L^2, L^2} \leq C \langle t \rangle^\beta, \quad t \in \mathbb{R}.$$

*Proof.* We denote by  $ad^l(H_\theta) = [x_i, \dots, [x_i, H_\theta]]$ , here  $[A, B] = AB - BA$ . We claim that for any  $l \in \mathbb{N}$ ,  $ad^l(R_\theta)$  is uniformly bounded on  $L^2$ . First, it's easy to see

$$[x_i, (-\Delta)^\alpha] = 2\alpha \partial_i (-\Delta)^{\alpha-1},$$

and

$$[x_i, [x_i, (-\Delta)^\alpha]] = 2\alpha(-\Delta)^{\alpha-1} + 4\alpha(\alpha-1)\partial_i^2(-\Delta)^{\alpha-2}.$$

Generally, we have

$$\begin{cases} ad^{2l-1}(H) &= \sum_{j=1}^l C_{\alpha,j} \partial_i^{2j-1} (-\Delta)^{\alpha-l+1-j}, & 2l-1 \leq \alpha, \\ ad^{2l}(H) &= \sum_{j=0}^l D_{\alpha,j} \partial_i^{2j} (-\Delta)^{\alpha-l-j}, & 2l \leq \alpha, \end{cases}$$

where  $C_{\alpha,j}, D_{\alpha,j}$  are constants only depending on  $j, \alpha$ . We note that  $ad^l(R)$  is a linear combination of such terms:

$$R ad^{i_1}(H) R ad^{i_2}(H) R \dots ad^{i_r}(H) R,$$

where  $1 \leq i_r \leq l$ ,  $1 \leq r \leq l$ , and  $\sum_{j=1}^r i_j = l$ . Since  $V \in K_{2\alpha}(\mathbb{R}^n)$ , and it is infinitesimally form bounded with respect to  $(-\Delta)^\alpha$  (see [40]), i.e., for any  $\varepsilon > 0$ , there exists some  $\lambda_\varepsilon > 0$ ,

$$\left| \int_{\mathbb{R}^n} V |f|^2 dx \right| \leq \varepsilon \|(-\Delta)^{\frac{\alpha}{2}} f\|_{L^2}^2 + \lambda_\varepsilon \|f\|_{L^2}^2,$$

then it can be checked that  $(-\Delta)^{\frac{\alpha}{2}} R^{\frac{1}{2}}$ ,  $\partial_i (-\Delta)^{\frac{\alpha-1}{2}} R^{\frac{1}{2}}$  are  $L^2$  bounded. Indeed, the first one is equivalent to  $\|(-\Delta)^{\frac{\alpha}{2}} f\|^2 \leq C \|(H+M)^{\frac{1}{2}} f\|^2$ ,  $f \in H^\alpha(\mathbb{R}^n)$ . Let  $\varepsilon = \frac{1}{2}$  in the inequality above, and choose  $M > \lambda_{\frac{1}{2}}$ , one has

$$((H+M)f, f) \geq \frac{1}{2} \|(-\Delta)^{\frac{\alpha}{2}} f\|_2^2 + (M - \lambda_{\frac{1}{2}}) \|f\|_2^2,$$

then we have  $(-\Delta)^{\frac{\alpha}{2}} R^{\frac{1}{2}}$  is  $L^2$  bounded. Note that  $\partial_i (-\Delta)^{\frac{\alpha-1}{2}} R^{\frac{1}{2}} = R_i (-\Delta)^{\frac{\alpha}{2}} R^{\frac{1}{2}}$ , where  $R_i$  is the Riesz transformation, which is clearly  $L^2$  bounded, so  $\partial_i (-\Delta)^{\frac{\alpha-1}{2}} R^{\frac{1}{2}}$  is also a bounded operator in  $L^2$ . Then it's not hard to see that  $R^{\frac{1}{2}} ad^{i_r}(H) R^{\frac{1}{2}}$  is  $L^2$  bounded, hence  $ad^l R$  is  $L^2$  bounded.

In order to prove the claim, we need to verify that the estimate

$$\|R_\theta ad^{i_1}(H_\theta) R_\theta ad^{i_2}(H_\theta) R_\theta \dots ad^{i_r}(H_\theta) R_\theta\|_{L^2, L^2} \leq C$$

is uniformly with respect to  $\theta$ , where  $R_\theta = (H_\theta + M)^{-1}$  and  $\theta \in (0, 1]$ . Without loss of generality, we can assume  $H_\theta \geq 0$ , then for any  $i_j$  with  $1 \leq j \leq r$ , let  $\alpha_j = \frac{2\alpha - i_j}{4\alpha} < \frac{1}{2}$  and  $R_{\theta, \alpha_j} = R_\theta^{\alpha_j} \text{ad}^{i_j}(H) R_\theta^{\alpha_j}$ , after scaling, we get

$$\begin{aligned} \|R_\theta^{\alpha_j} \text{ad}^{i_j}(H_\theta) R_\theta^{\alpha_j}\|_{L^2, L^2} &\leq \|(\theta H + M)^{-\alpha_j} \theta^{\frac{2\alpha - i_j}{2\alpha}} \text{ad}^{i_j}(H) (\theta H + M)^{-\alpha_j}\|_{L^2, L^2} \\ &\leq \|(H + M)^{-\alpha_j} \text{ad}^{i_j}(H) (H + M)^{-\alpha_j}\|_{L^2, L^2}. \end{aligned}$$

It's clear that  $\|R_\theta^t\|_{L^2, L^2} \leq M^{-t}$ ,  $0 \leq t \leq 1$ . So

$$\begin{aligned} &\|R_\theta \text{ad}^{i_1}(H_\theta) R_\theta \text{ad}^{i_2}(H_\theta) R_\theta \dots \text{ad}^{i_r}(H_\theta) R_\theta\|_{L^2, L^2} \\ &= \|R_\theta^{1-\alpha_1} R_{\theta, \alpha_1} R_\theta^{1-\alpha_1-\alpha_2} R_{\theta, \alpha_2} \dots R_{\theta, \alpha_r} R_\theta^{1-\alpha_r}\|_{L^2, L^2} \leq C, \end{aligned}$$

which proves the claim.

Note that

$$\frac{d}{ds}(e^{-isR_\theta} x_i e^{-i(t-s)R_\theta}) = -ie^{-isR_\theta} \text{ad}(R_\theta) e^{-i(t-s)R_\theta},$$

we have

$$\text{ad}^l(e^{-itR_\theta}) = -i \int_0^t e^{-isR_\theta} \text{ad}(R_\theta) e^{-i(t-s)R_\theta} ds,$$

using this fact and the claim above repeatedly, we get

$$\|\text{ad}^l(e^{-itR_\theta})\|_{L^2, L^2} < C \langle t \rangle^l, \quad t \in \mathbb{R}.$$

Then, according to Lemma 3.1 in [26], one obtains the theorem first for integer and then for any  $\beta > 0$  by Carderon-Lions interpolation theorem.  $\square$

To proceed, we also need the following result in [25] concerning the boundedness of operators on  $l^1(L^2)$ , if  $A \in \mathcal{A}_\beta$  for some  $\beta > \frac{n}{2}$ , then  $A$  is bounded on  $l^1(L^2)$ , here

$$\mathcal{A}_\beta = \{A \in B(L^2) : \sup_{k \in \mathbb{Z}^n} \|\langle \cdot - k \rangle^\beta A \chi_{C(k)} \varphi\| \leq C \|\varphi\|\},$$

here and below,  $\chi_\Omega$  is the characteristic function of the set  $\Omega$ . More specifically, for  $A \in \mathcal{A}_\beta$ , if we write  $\|A\|_\beta = \sup_{k \in \mathbb{Z}^n} \|\langle \cdot - k \rangle^\beta A \chi_{C(k)} \varphi\|$ , then

$$(4.3) \quad \|A\|_{l^1(L^2), l^1(L^2)} \leq C \|A\|_\beta^{\frac{n}{2\beta}} \|A\|^{1-\frac{n}{2\beta}}.$$

*The proof of Theorem 1.2.* Firstly, we show that if  $f \in C_0^\infty(\mathbb{R})$ , then  $f(R_\theta)$  is uniformly bounded on  $l^1(L^2)$ .

In order to prove this, we note that followed by functional calculus, one has

$$f(R_\theta) = (2\pi)^{\frac{1}{2}} \int e^{-itR_\theta} \widehat{f}(t) dt,$$

then, it follows from Theorem 4.2 that

$$\begin{aligned} \|\langle \cdot - k \rangle^\beta f(R_\theta) \langle \cdot - k \rangle^{-\beta}\| &\leq (2\pi)^{\frac{1}{2}} \int \|\langle \cdot - k \rangle^\beta e^{itR_\theta} \langle \cdot - k \rangle^{-\beta}\| |\widehat{f}(t)| dt \\ &\leq C \int |\widehat{f}(t)| \langle t \rangle^\beta dt, \end{aligned}$$

where  $C$  is independent of  $\theta$ . Note that

$$\|\langle \cdot - k \rangle^\beta f(R_\theta) \chi_{C(k)}\| \leq (1 + |k|)^{\frac{\beta}{2}} \|\langle \cdot - k \rangle^\beta f(R_\theta) \langle \cdot - k \rangle^{-\beta}\|,$$

we obtain by definition

$$(4.4) \quad \|f(R_\theta)\|_\beta \leq C \int |\widehat{f}(t)| \langle t \rangle^\beta dt,$$

now the assertion follows from (4.3).

Next, we claim that If  $f \in C_0^\infty(\mathbb{R})$ , then  $e^{-itH_\theta} f(H_\theta)$  is bounded on  $l^1(L^2)$ . Furthermore, for all  $\gamma > \frac{n}{2}$ , we have

$$\|e^{-itH_\theta} f(H_\theta)\|_{l^1(L^2), l^1(L^2)} \leq C_\gamma \langle t \rangle^\gamma,$$

where  $C_\gamma$  depends only on  $\gamma$ .

Indeed, for bounded function  $g(\mu)$ , we write  $g_t(\mu) = e^{it(M-\frac{1}{\mu})} g(\mu)$ ,  $t, \mu \in \mathbb{R}$ , then by functional calculus,

$$g_t(R) = e^{-itH} g(R), \quad g \in C_0^\infty(\mathbb{R}).$$

Applying Schwartz's inequality yields

$$(4.5) \quad \begin{aligned} \int |\widehat{g}_t(s)| \langle s \rangle^\beta ds &\leq \|\langle s \rangle^m \widehat{g}_t(s)\|_2 \|\langle s \rangle^{\beta-m}\|_2 \\ &\leq C(\|g_t\|_2 + \|g_t^{(m)}\|_2) \\ &\leq C \langle t \rangle^m, \end{aligned}$$

provided  $\beta < m - \frac{1}{2}$ . Note that there exists  $g \in C_0^\infty(\mathbb{R})$ , such that  $g(\frac{1}{\lambda+M}) = f(\lambda)$ ,  $\lambda \in \sigma(H)$ , then based on (4.4) and (4.5), we get

$$\|e^{-itH_\theta} f(H_\theta)\|_\beta = \|g_t(R_\theta)\|_\beta \leq C \langle t \rangle^m.$$

In view of (4.3), we have

$$\|e^{-itH_\theta} f(H_\theta)\|_{l^1(L^2), l^1(L^2)} \leq C \langle t \rangle^{\frac{n}{2\beta}m}.$$

Now for any fixed  $\gamma > \frac{n}{2}$ , we can choose sufficiently large numbers  $\beta, m$  with  $\frac{n}{2} < \beta < m - \frac{1}{2}$  such that  $\frac{nm}{2\beta} < \gamma$ , which implies the claim.

Now we use the claim above to show that if  $g \in C_0^\infty(\mathbb{R})$ , then

$$(4.6) \quad \|g(\theta H) e^{-it\theta H}\|_{L^p, L^p} \leq C \langle t \rangle^\beta$$

holds uniformly with  $\theta \in (0, 1]$ ,  $1 \leq p \leq \infty$ ,  $\beta > n|\frac{1}{2} - \frac{1}{p}|$ . In addition, it's valid for  $g$  running over a bounded set of  $C_0^\infty(\mathbb{R})$ .

After a scaling argument (see [25]), it suffices to show that

$$\|g(H_\theta) e^{-itH_\theta}\|_{L^p, L^p} \leq C \langle t \rangle^\beta$$

holds uniformly with  $\theta \in (0, 1]$ .

Actually, according to theorem 4.1, when  $\beta > \frac{n}{4m}$ ,  $(H_\theta + M)^{-\beta}$  is uniformly bounded from  $L^1$  to  $l^1(L^2)$ , which is imbedded in  $L^1$ , so by the claim we have just proved above, we have

$$\begin{aligned} \|e^{-itH_\theta} g(H_\theta)\|_{L^1, L^1} &\leq C \|e^{-itH_\theta} h(H_\theta)\|_{l^1(L^2), l^1(L^2)} \|H_\theta + M\|_{L^1, l^1(L^2)}^{-\beta} \\ &\leq C \langle t \rangle^\gamma, \end{aligned}$$

here  $h(x) = g(x)(x+M)^\beta$ . The general case follows by duality and an interpolation argument.

At last, we shall see that uniform estimates (4.6) and a dyadic decomposition imply our main result. Indeed, let  $\varphi \in C_0^\infty(-2, 2)$ ,  $\varphi = 1$  on  $(-\frac{1}{2}, \frac{1}{2})$ , such that

$$\sum_0^\infty \varphi_k(\lambda) = 1$$

where  $\varphi_k(\lambda) = \varphi(2^{-k}\lambda)$ ,  $k = 1, 2, \dots$ . Set  $f(\lambda) = (\lambda + M)^{-\beta}$  and  $f_k(\lambda) = \varphi(\lambda) \cdot f(2^k\lambda)$  for  $k \geq 0$ . Then we have  $f(\lambda) = \sum_0^\infty f_k(2^{-k}\lambda)$ ,  $\lambda > 0$ , and  $\text{supp } f_k \subseteq (\frac{1}{2}, 2)$ ,  $k = 1, 2, \dots$ . Note that

$$|f_k^{(\alpha)}(\lambda)| \leq C 2^{k\alpha} \langle 2^k \lambda \rangle^{-\beta-\alpha} \leq C 2^{-\beta k}, \quad k \geq 0,$$

so  $\{2^{\beta k} f_k(\lambda)\}_{k=0}^\infty$  are bounded in  $C_0^\infty(\mathbb{R})$ .

Fixed  $\beta > n|\frac{1}{2} - \frac{1}{p}|$ , it suffices to show Theorem 1.2 for  $\gamma \in (n_p, \beta)$ , since for large  $\gamma$  can be reduced to this case. It follows from (4.6) that

$$\|2^{\beta k} e^{-it\theta H} f_k(\theta H)\|_{L^p, L^p} \leq C \langle t \rangle^\gamma,$$

where  $k \geq 0$ ,  $\theta \in (0, 1]$ . Now set  $\theta = \frac{1}{2^k}$ ,  $t = 2^k s$

$$\|e^{-isH} f_k(\frac{1}{2^k} H)\|_{L^p, L^p} \leq C 2^{-(\beta-\gamma)k} \langle t \rangle^\gamma,$$

therefore

$$\begin{aligned} \|e^{-isH} f(H)\|_{L^p, L^p} &\leq \sum_{k=0}^\infty \|e^{-isH} f_k(2^{-k} H)\|_{L^p-L^p} \\ &\leq C \sum_{k=0}^\infty 2^{-(\beta-\gamma)k} \langle s \rangle^\gamma \\ &\leq C \langle s \rangle^\gamma. \end{aligned}$$

#### ACKNOWLEDGEMENTS

Zheng was supported by the National Natural Science Foundation of China (No. 11471129). Wang was Supported by the NSFC (No. 11426209), and the Fundamental Research Funds for the Central Universities, China University of Geosciences(Wuhan), Grant No. CUGL140838.

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